Journal of Statistical Physics, Vol. 79, Nos. 3/4, 1995

The Invariant Densities for Maps Modeling Intermittency

Maximilian Thaler[1]

Received August 17, 1994

For a one-parameter family of maps modeling intermittency the explicit formula of the invariant density is presented.

KEY WORDS: Intermittency; maps of the interval; invariant density.

In ref. 2 the maps

$$Tx = (1+\varepsilon)\,x + (1-\varepsilon)\,x^2 \pmod 1 \qquad (0 \leqslant \varepsilon \leqslant 2)$$

are used as simplified models to discuss the phenomenon of intermittency. Quoting the author, intermittency is the 'seemingly random alternation of long regular or laminar phases and short irregular or turbulent bursts.' The above maps display this phenomenon if ε (or $2-\varepsilon$) is small, and in very strong form if $\varepsilon = 0$ (or $\varepsilon = 2$).

A central tool in the analysis of the dynamics of such maps is the invariant measure. It follows from standard results (see, e.g., refs. 1 and 3) that for each $\varepsilon \in [0, 2]$ the map T has a unique invariant measure μ equivalent to Lebesgue measure, which is finite if $0 < \varepsilon < 2$ and infinite if $\varepsilon = 0$ or $\varepsilon = 2$. In fact, it follows that the invariant density $d\mu/d\lambda$ has a version h which is continuous (even analytic) and positive on $[0, 1]$ if $0 < \varepsilon < 2$, and takes the form $h_0(x)/x$, resp. $h_0(x)/(1-x)$, in case $\varepsilon = 0$, resp. $\varepsilon = 2$, where h_0 is continuous and positive on $[0, 1]$. The question now is whether one can find these densities explicitly.

In ref. 2, using analytic arguments, the author obtains the term

$$\frac{c}{\varepsilon + (1-\varepsilon)\,x} \qquad (c \text{ constant})$$

[1] Institute of Mathematics, University of Salzburg, 5020 Salzburg, Austria. E-mail: THALER@EDVZ.SBG.AC.AT.

0022-4715/95/0500-0739$07.50/0 © 1995 Plenum Publishing Corporation

as an approximation to the invariant density for small ε and x. A more intuitive version of this argument yielding the same approximation could be as follows.

Let h be as above, let ε and x be small, and let $A = [x, 1[$. By the stationarity of μ,

$$\mu(A \cap T^{-1}A^c) = \mu(A^c \cap T^{-1}A)$$

i.e., the amount of mass in A transported to A^c in one step is the same as that in A^c transported to A. Note that T has two full branches. Let α denote the point of discontinuity, and let u_0, resp. u_1, denote the inverse function of T restricted to $[0, \alpha[$, resp. $[\alpha, 1[$. Then,

$$\mu(A \cap T^{-1}A^c) = \mu([\alpha, u_1(x)[) \approx h(\alpha)\, u_1'(0) \cdot x$$

and

$$\mu(A^c \cap T^{-1}A) = \mu([u_0(x), x[) \approx h(x)(x - u_0(x))$$
$$\approx h(x) \frac{Tx - x}{1 + \varepsilon}$$

and hence

$$h(x) \approx \frac{cx}{Tx - x} = \frac{c}{\varepsilon + (1 - \varepsilon)\, x}$$

The purpose of this note is to communicate the following result.

Proposition. For each $\varepsilon \in [0, 2]$,

$$h(x) = \frac{1}{\varepsilon + (1 - \varepsilon)\, x} + \frac{1}{1 + (1 - \varepsilon)\, x}, \qquad x \in [0, 1[$$

Proof. The straightforward method of proof is to verify Kuzmin's equation

$$h = h \circ u_0 \cdot u_0' + h \circ u_1 \cdot u_1'$$

by calculating the functions u_0, u_1 explicitly. In order to give some insight into the process of finding h, we proceed in a different way.

Let $g(x) = (1 + \varepsilon)\, x + (1 - \varepsilon)\, x^2$, $x \in [0, 1[$, and let h be a nonnegative measurable function on $[0, 1[$. Define the function ψ on $[0, 2[$ by

$$\psi = h \circ g^{-1} \cdot (g^{-1})'$$

so that

$$h = \psi \circ g \cdot g' \quad \text{on} \quad [0, 1[$$

Since

$$g(u_0(x)) = x \qquad \text{and} \qquad g(u_1(x)) = x + 1$$

we have

$$h(u_0(x))\, u_0'(x) = \psi(x)$$

and

$$h(u_1(x))\, u_1'(x) = \psi(x+1) \qquad \text{for} \quad x \in [0, 1[$$

Therefore h is a solution of the Kuzmin equation if and only if

$$\psi(g(x)) \cdot g'(x) = \psi(x) + \psi(x+1) \tag{$*$}$$

Thus we see that h is an invariant density for T if and only if

$$h(x) = \psi(x) + \psi(x+1)$$

where ψ is a nonnegative measurable function on $[0, 2[$ satisfying ($*$). (Equation ($*$) is just the Kuzmin equation for the conjugated map $S = g \circ T \circ g^{-1}$ on $[0, 2[$.)

The approximation resulting from the heuristic considerations suggests that we start the trial and error procedure to find the function ψ with $\psi(x) = 1/[\varepsilon + (1-\varepsilon)\, x]$. We have

$$\varepsilon + (1-\varepsilon)\, g(x) = [\varepsilon + (1-\varepsilon)\, x][\varepsilon + (1-\varepsilon)(x+1)]$$

and logarithmic differentiation shows that this choice of ψ is in fact the correct one. ∎

REFERENCES

1. M. Halfant, Analytic properties of Rényi's invariant density, *Isr. J. Math.* **27**:1–20 (1977).
2. P. Manneville, Intermittency, self-similarity and $1/f$ spectrum in dissipative dynamical systems, *J. Phys.* (Paris) **41**:1235–1243 (1980).
3. M. Thaler, Transformations on [0, 1] with infinite invariant measures, *Isr. J. Math.* **46**:67–96 (1983).

Communicated by J. L. Lebowitz